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Spinning Strings, Black Holes and Stable Closed Timelike Geodesics

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Abstract The existence and stability under linear perturbation of closed timelike curves in the spacetime associated to Schwarzschild black hole pierced by a spinning string are studied. Due to the superposition of the black hole, we find that the spinning string spacetime is deformed in such a way to allow the existence of closed timelike geodesics.

Keywords Closed timelike geodesics · Linear stability · Time machines · Black holes · Cosmic strings · Torsion lines

The existence of closed timelike curves (CTCs) in the Gödel universe and other apacetimes is a worrying fact since these curves show a clear violation of causality. In some cases these CTCs can be disregarded by energy considerations. Their existence requires an external force acting along the whole CTC, process that may consume a great amount of energy. The energy needed to travel along a CTC in Gödel's universe is computed in [1]. When the external force is null the energy needed to travel is also null. Therefore, in principle, the existence of closed timelike geodesics (CTGs) presents a bigger problem of breakdown of causality.

The classical problem of the existence of closed geodesics in Riemannian geometry was solved by Hadamard [2] in two dimensions and by Cartan [3] in an arbitrary number of dimensions. As a topological problem, the existence of CTGs was proved by Tipler [4] in a class of four-dimensional compact Lorentz manifolds with covering space containing a compact Cauchy surface. In a compact pseudo-Riemannian manifold with Lorentzian signature (Lorentzian manifold) Galloway [5] found sufficient conditions to have CTGs, see also [6].

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To the best of our knowledge there are four solution to the Einstein equations generated by matter with positive mass density that contain CTGs: (a) Soares [7] found a class of cosmological models, solutions of Einstein-Maxwell equations, with a subclass where the timelike paths of matter are closed. For these models the existence of CTGs is demonstrated and explicit examples are given. These CTGs are not linearly stable [8]. (b) Steadman [9] described the behavior of CTGs in a vacuum exterior for the van Stockum solution that represents an infinite rotating dust cylinder. For this solution explicit examples of CTCs and CTGs are shown. There are stable CTGs in this spacetime [8]. (c) Bonnor and Steadman [10] studied the existence of CTGs in a spacetime with two spinning particles each one with magnetic moment equal to angular moment and mass equal to charge (Perjeons), in particular, they present a explicit CTG. This particular CTG is not stable, but there exist many other that are stable [11]. (d) There are linearly stable CTGs [8] in one of the Gödel-type metrics with not flat background studied by Gürses et al. [12, 13]. For CTGs in a spacetime associated to a cloud of strings with negative mass density see [14]. These CTGs are not stable [8].

The existence of CTCs in a spacetime whose source is a spinning string has been investigated by many authors (see for instance [15–19]). The interpretation of these strings as torsion line defects can be found in [20, 21], see also [22, 23]. These torsion line defects appear when one tries to stabilize two rotating black holes kept apart by spin repulsion [24]. Also, the black hole thermodynamics associated to a static black hole pierced by a non rotating string was studied some time ago by Aryal et al. [25].

In the present work we study the existence and stability of CTCs under linear perturbations in the spacetime associated to Schwarzschild black hole (BH) pierced by a spinning string. Even though this spacetime is more a mathematical curiosity than an example of a real spacetime we believe that the study of stability of CTCs and CTGs can shed some light into the existence of this rather pathological curves. In particular, we study sufficient conditions to have linearly stable CTGs. We find that these conditions are not very restrictive and can be easily satisfied. Furthermore, we compared them with the same conditions studied by Galloway [5] for a compact Lorentzian manifold.

Let us consider the spacetime with metric,

$$ds^2 = \left(1 - \frac{2m}{r}\right)(dt - \alpha d\varphi)^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \beta^2 \sin^2\theta d\varphi^2), \tag{1}$$

where $\alpha = 4S$ and S is the string's spin angular momentum per unit of length, $\beta = 1 - 4\lambda$ and λ is the string's linear mass density that is equal to its tension ($\lambda \leq 1/4$).

In the particular case, $\alpha = 0$ and $\beta = 1$, the metric (1) reduces to the Schwarzschild solution. When $m = 0$, (1) represents a spinning string, with the further specialization $\beta = 1$ (not deficit angle) we have a pure massless torsion line defect [20, 21]. Therefore the metric (1) can be considered as representing the spacetime associated to a Schwarzschild black hole pierced by a spinning string.

Let us denote by γ a closed curve given in its parametric form by,

$$t = t_0, \qquad r = r_0, \qquad \varphi \in [0, 2\pi], \qquad \theta = \frac{\pi}{2}, \tag{2}$$

where t_0 and r_0 are constants. When γ is parametrized with an arbitrary parameter σ, we have a timelike curve when $\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma} > 0$. This condition reduces to $g_{\varphi\varphi} > 0$, i.e.,

$$(1 - 2m/r_0)\alpha^2 - r_0^2\beta^2 > 0. \tag{3}$$

A generic CTC γ satisfies the system of equations given by

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = F^\mu(x), \tag{4}$$

where the overdot indicates derivation with respect to s, $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols and F^μ is a specific external force ($a^\mu = F^\mu$). The nonzero component of the four-acceleration of γ is

$$a^r = \frac{1}{r_0^3}(r_0 - 2m)(\alpha^2 m - r_0^3\beta^2)\dot{\varphi}^2. \tag{5}$$

Our goal is to study the behavior of closed timelike geodesics. Therefore taking α as one of the two solutions of

$$\alpha^2 m - r_0^3\beta^2 = 0, \tag{6}$$

we have $a^r = 0$. Under this condition (3) is satisfied when $r_0 > 3m$, that put the CTG outside the black hole.

Let $\tilde{\gamma}$ be the curve obtained from γ after a small perturbation ξ, i.e., $\tilde{x}^\mu = x^\mu + \xi^\mu$. From (4) one finds that the system of differential equations satisfied by the perturbation ξ is [26],

$$\frac{d^2\xi^\alpha}{ds^2} + 2\Gamma^\alpha_{\beta\mu}\frac{d\xi^\beta}{ds}u^\mu + \Gamma^\alpha_{\beta\mu,\lambda}\xi^\lambda u^\beta u^\mu = F^\alpha_{,\lambda}\xi^\lambda. \tag{7}$$

For the above mentioned closed timelike geodesic the system (7) reduces to

$$\ddot{\xi}^0 + k_1\dot{\xi}^1 = 0, \tag{8}$$

$$\ddot{\xi}^1 + k_2\dot{\xi}^0 + k_3\xi^1 = 0, \tag{9}$$

$$\ddot{\xi}^2 + k_4\dot{\xi}^1 = 0, \tag{10}$$

$$\ddot{\xi}^3 + k_5\xi^3 = 0, \tag{11}$$

where

$$\begin{aligned} &k_1 = 2\Gamma^0_{21}\dot{\varphi}, \qquad k_2 = 2\Gamma^1_{20}\dot{\varphi}, \qquad k_3 = \Gamma^1_{22,1}\dot{\varphi}^2, \\ &k_4 = 2\Gamma^2_{21}\dot{\varphi}, \qquad k_5 = \Gamma^3_{22,3}\dot{\varphi}^2. \end{aligned} \tag{12}$$

A curve γ parametrized by the proper time, s, is timelike when $\dot{x}^\mu\dot{x}_\mu = 1$. For the curve $\gamma(s)$ we have that this last condition gives us,

$$\dot{\varphi}^2 = \frac{m}{\beta^2 r_0^2(r_0 - 3m)}. \tag{13}$$

The solution of (9)–(11) is

$$\begin{aligned} &\xi^0 = -k_1(c_3\sin(\omega s + c_4)/\omega + \lambda s) + c_1 s + c_5, \\ &\xi^1 = c_3\cos(\omega s + c_4) + \lambda, \\ &\xi^2 = -k_4(c_3\sin(\omega s + c_4)/\omega + \lambda s) + c_2 s + c_6, \\ &\xi^3 = c_7\cos(\sqrt{k_5}s + c_8), \end{aligned} \tag{14}$$

where $c_i,\ i = 1, \ldots, 8$ are integration constants,

$$\omega = \sqrt{k_3 - k_1 k_2} = [\beta^2 (r_0 - 6m)\dot{\varphi}^2 / r_0]^{1/2}, \tag{15}$$

and $\lambda = -k_2 c_1/\omega^2$. Thus when $r_0 > 6m$, the constant ω is real and the solution (14) shows the typical behavior for stability, i.e., vibrational modes untangled with translational ones that can be eliminated by a suitable choice of the initial conditions.

When the black hole is removed, we are left with the spacetime of the spinning string whose line element is,

$$ds^2 = (dt - \alpha d\varphi)^2 - dr^2 - r^2 (d\theta^2 + \beta^2 \sin^2\theta d\varphi^2). \tag{16}$$

The closed curve, γ, is timelike when $\alpha^2 - r_0^2\beta^2 > 0$. The a^r-component of the four-acceleration is given by $a^r = -\beta^2 r_0 \dot{\varphi}^2$. Thus for $r < |\alpha/\beta|$ we have closed timelike curves, which are not geodesics.

For the closed curve (2) the system (7) is written now as in (9)–(11) replacing (10) by

$$\ddot{\xi}^1 + k_2 \dot{\xi}^2 + k_3 \xi^1 = \partial_r (\Gamma^1_{22}\dot{\varphi}^2)\xi^1, \tag{17}$$

where now $k_2 = 2\Gamma^1_{22}\dot{\varphi}$ and $\dot{\varphi}^2 = (\alpha^2 - r_0^2\beta^2)^{-1}$. In this particular case the solution of (7) has the same form that (14) with $\omega^2 = 2\beta^2\dot{\varphi}^2(2 + \beta^2 r_0^2 \dot{\varphi}^2)$. Therefore, the CTCs are stable.

In summary, there exist linearly stable CTCs in the spacetime related to a spinning string and these curves are restricted to a small region of the spacetime. Closed timelike geodesics do not exist in this spacetime.

For the nonlinear superposition of a spinning string with a Schwarzschild black hole the new spacetime has linearly stable CTGs. The region of stability is the same of the usual circular geodesics in the Schwarzschild black hole alone. The presence of the spinning string does not affect the stability of the orbits. It seems that torsion lines defects superposed to matter (not strings, $\beta = 1$) is a main ingredient to have stable CTGs. Loosely speaking, we have that a torsion line defect alone makes possible the existence of CTCs. When the black hole is present the spinning string spacetime is deformed in such a way to allow the existence of a CTG. This fact is also confirmed in the case of the two Perjeons solutions studied in [10] wherein the torsion line defect is a main ingredient to have CTCs and CTGs.

It is instructive to look the previous results in a more direct and graphic way. The length of CTC in (2) only depends on the value of $r = r_0$. We find,

$$s(r_0) = 2\pi\sqrt{g_{\varphi\varphi}(r_0)}, = 2\pi[(1 - 2m/r_0)\alpha^2 - r_0^2\beta^2]^{1/2}. \tag{18}$$

This function has a local maximum for

$$r_m = (m\alpha^2/\beta^2)^{1/3}. \tag{19}$$

Note that this equation is equivalent to (6), the condition to have a geodesic.

The role of the black hole mass, in the appearance of CTGs, is to produce a local maximum in the length function, $s(r_0)$. This maximum gives us the position of the CTG that in our case is located outside of the source of the spacetime, beyond the black hole horizon.

In Fig. 1 we present, as a solid line the function $s(r_0)$ for a spinning string, and as a dashed line the same function for the superposition of the black hole with the previously

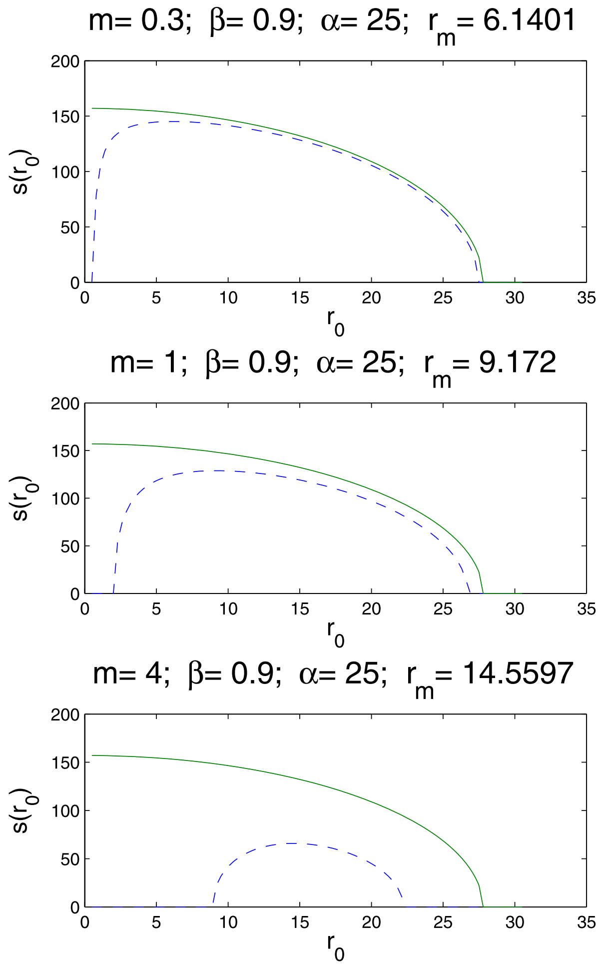


Fig. 1 The function $s(r_0)$ for a spinning string (*solid line*) and for a black hole pierced by the string (*dashed line*). We see how the presence of the mass shift the maximum of $s(r_0)$ for the string that is located at $r_0 = 0$ to a position outside the black hole horizon. The maximum, r_m represent the radius of the CTGs, the first two are stable and the second is not

mentioned string for the same values of the parameters $\alpha = 25$ (spin parameter) and deficit angle parameter $\beta = 0.9$ and different values of the black hole mass ($m = 0.3, 1, 4$). We see how the presence of the mass shift the maximum for the string located at $r_0 = 0$ to a position $r_0 > 3m$. Also the points under the curves represent the pairs $[r_0, s(r_0)]$ for CTCs in each case. We note that the region for CTCs for the black hole pierced by the string diminishes when the mass increases. The maximum of the dashed line represents the CTG. We see, that in the first two cases the CTGs are stable ($r_m > 6m$) and in the last case the CTG is not stable ($r_m < 6m$).

In Fig. 2 we keep the value of the black hole mass constant, $m = 1$, as well as, the spin parameter, $\alpha = 25$, and change the deficit angle parameter $\beta = 0.4, 0.7, 0.9$. We see that the larger the string density, $\lambda = (1 - \beta)/4$, the larger the region for CTCs.

In Fig. 3 we keep the value of the black hole mass constant, $m = 1$, as well as, the deficit angle parameter $\beta = 0.9$ and change the spin parameter $\alpha = 15, 20, 25$. We see that the regions where the CTCs appear are larger for bigger spin parameter. This parameter is essential to have CTCs and CTGs in this case.

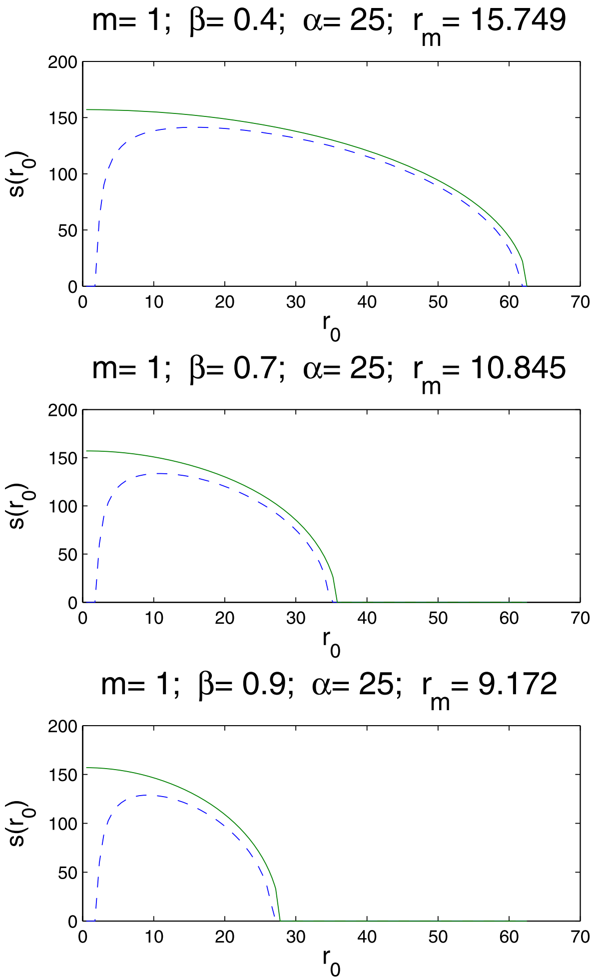
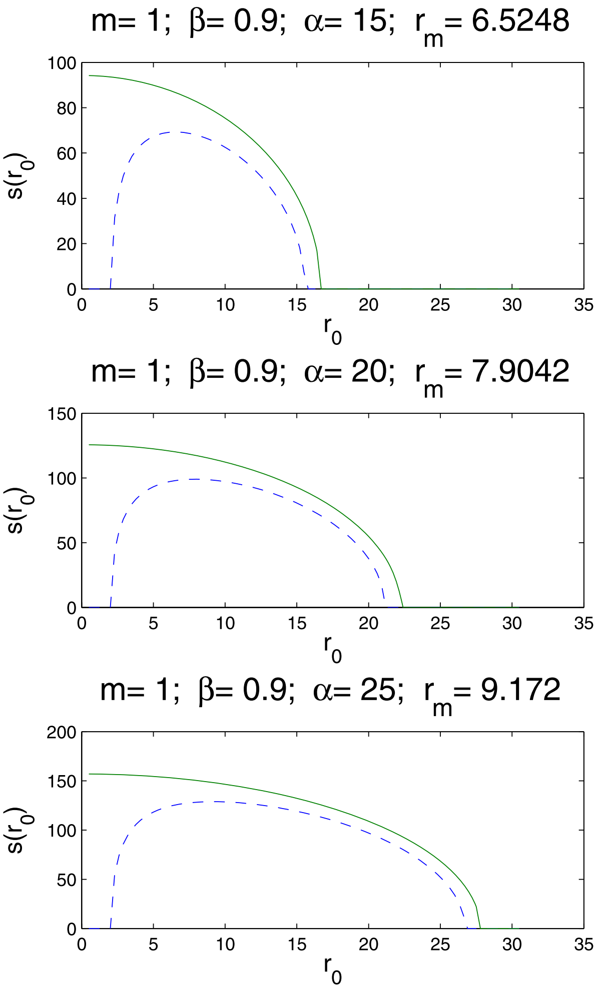


Fig. 2 The function $s(r_0)$ for a spinning string (*solid line*) and for a black hole pierced by the string (*dashed line*). We see how the size of the deficit angle parameter β changes the region for CTCs and the value of r_m

As we said before the existence of a CTGs does not put restrictions on the energy to travel along this curve. Furthermore, the force needed to move near a stable geodesic is small. Therefore, the energy required will be also small. In principle this small force can be provided by and engine, say a rocket. Hence there will be not a severe energy restriction to travel near to a geodesic. Furthermore, when moving along a stable CTG the control problem is a trivial one. Small trajectory corrections require small energy, also we do not have the danger to enter into a run away situation.

A result from Galloway [5] states that in a compact Lorentzian manifold, each stable free t-homotopy class contains a longest closed timelike curve, and this curve is necessarily a closed timelike geodesic. The assumption that M be compact can be weakened, it is sufficient to assume that there exists an open set U in M with compact closure such that each curve $\gamma \in \mathcal{C}$ (the free t-homotopy class) is contained in U. In our case th Gödel universe and other apacetimes e region containing the CTCs in $\mathcal{C}$ is not compact. Therefore Galloway's conditions do not apply in this case, they are too strong.

Fig. 3 The function $s(r_0)$ for a spinning string (*solid line*) and for a black hole pierced by the string (*dashed line*). We see how the size of the spin parameter α changes the region for CTCs and the value of r_m. The spin parameter, in this case, is the essential ingredient to have CTCs and CTGs



We want to point out that the stability of the circular orbits does not depend on the fact of the orbit be a CTG. We found the same region of stability of the usual circular geodesics. This result is not surprising since our pierced black hole is locally identical to a usual black hole. Moreover one can consider black holes surrounded by different axially symmetric distributions of matter [27] pierced by a spinning string. In this case, depending on the different parameters of the solution, we can also have CTGs and their stability will be the same as the usual circular orbits considered in [27].

Furthermore, we analyze if the CTGs studied in the present work satisfy the sufficient conditions of Galloway's theorem for the existence of CTGs. We found that ours CTGs do not satisfy these conditions. The possibility of an example that satisfy exactly the conditions of this theorem is under study. We want to mention that the solution of Einstein equations considered in this work is much simpler that the ones listed in the introduction.

Finally, we notice that the spacetime associated to the black hole pierced by a spinning string is not a counter example to the Chronology Protection Conjecture [28] that essentially says that the laws of the physics do not allow the appearance of closed timelike curves. A valid dynamic to built this spacetime is not known.

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References

1. Pfarr, J.: Gen. Relativ. Gravit. **13**, 1073 (1981)
2. Hadamard, J.: J. Math. Pures Appl. **4**, 27 (1896)
3. Cartan, E.: Leçons sur la géométrie de Riemann. Gauthier-Villars, Paris (1928)
4. Tipler, F.J.: Proc. Am. Math. Soc. **76**, 145 (1979)
5. Galloway, G.J.: Trans. Am. Math. Soc. **285**, 379 (1984)
6. Guerdiri, M.: Trans. Am. Math. Soc. **359**, 2663 (2007)
7. Soares, I.D.: J. Math. Phys. **21**, 591 (1980)
8. Rosa, V.M., Letelier, P.S.: Linear stability of closed timelike geodesics. In: Scherfner, M. (ed.) Gödel-type Spacetimes: History and New Developments, in press. 0706.3212 [gr-qc]
9. Steadman, B.R.: Gen. Relativ. Gravit. **35**, 1721 (2003)
10. Bonnor, W.B., Steadman, B.R.: Gen. Relativ. Gravit. **37**, 1833 (2005)
11. Rosa, V.M., Letelier, P.S.: Phys. Lett. A **370**, 99 (2007)
12. Gürses, M., Karasu, A., Sarioglu, O.: Class. Quantum Gravity **22**, 1527 (2005)
13. Gleiser, R.J., Gürses, M., Karasu, A., Sarroğlu, Ö.: Class. Quantum Gravity **23**, 2653 (2006)
14. Grøn, Ø., Johannesen, S.: New J. Phys. **10**, 103025 (2008). gr-qc/0703139
15. Deser, S., Jackiw, R., 't Hooft, G.: Phys. Rev. Lett. **68**, 267 (1992)
16. Cho, Y.M., Park, D.H.: Phys. Rev. D **46**, R1219 (1992)
17. Gott, J.R.: Phys. Rev. Lett. **66**, 1126 (1991)
18. Jensen, B., Soleng, H.H.: Phys. Rev. D **45**, 3528 (1992)
19. Soleng, H.H.: Phys. Rev. D **49**, 1124 (1994)
20. Letelier, P.S.: Class. Quantum Gravity **12**, 471 (1995)
21. Letelier, P.S.: Class. Quantum Gravity **12**, 2221 (1995)
22. de Sousa, P.: Nucl. Phys. B **346**, 440 (1990)
23. Petti, R.J.: Gen. Relativ. Gravit. **18**, 1124 (1986)
24. Letelier, P.S., Oliveira, S.R.: Phys. Lett. A **238**, 101 (1998)
25. Aryal, M., Ford, L.H., Vilenkin, A.: Phys. Rev. D **34**, 2263 (1986)
26. Rosa, V.M., Letelier, P.S.: Gen. Relativ. Gravit. **39**, 1419 (2007). gr-qc/0703100
27. Letelier, P.S.: Phys. Rev. D **68**, 104002 (2003)
28. Hawking, S.: Phys. Rev. D **46**, 603 (1991)